Distribution of Eigenvalues in Non-Hermitian Anderson Model

Ilya Ya. Goldsheid and Boris A. Khoruzhenko
School of Mathematical Sciences, Queen Mary & Westfield College,
University of London, London E1 4NS, U.K.
(22 July 1997)

We develop a theory which describes the behaviour of eigenvalues of a class of one-dimensional random non-Hermitian operators introduced recently by Hatano and Nelson. Under general assumptions on random parameters we prove that the eigenvalues are distributed along a curve in the complex plane. An equation for the curve is derived and the density of complex eigenvalues is found in terms of spectral characteristics of a "reference" hermitian disordered system. Coexistence of the real and complex parts in the spectrum and other generic properties of the eigenvalue distribution for the non-Hermitian problem are discussed.

PACS numbers: 72.15Rn, 74.60Ge, 05.45.+b

1. Complex eigenvalues of non-Hermitian random Hamiltonians have recently attracted much interest across several areas of physics [1–5]. However, the actual progress in understanding the statistics of such eigenvalues is mostly limited to the so-called 'zero-dimensional' case, i.e. to random matrices with no spatial structure. In this case a number of models can be treated analytically and their basic features are relatively well understood, although on different levels of rigour (see, for instance, recent works [4–6] and references therein). In contrast, little is known about spectra of non-Hermitian Hamiltonians in one or more dimensions. One of the challenging problems here involves an unusual localisation-delocalisation transition predicted by Hatano and Nelson [1].

Motivated by the studies of statistical mechanics of the magnetic flux lines in superconductors with columnar defects, Hatano and Nelson considered a model described by a random Schrödinger operator with a constant imaginary vector potential. Appealing to a qualitative reasoning they argued that already in one dimension some localised states undergo a delocalisation transition when the magnitude of potential increases. The eigenvalues corresponding to the localised states are real and those corresponding to the extended states are complex. The results of numerical calculations presented in [1] support these conclusions. They also show a surprising feature of the eigenvalue distribution in the model: the eigenvalues are attracted to a curve in the complex plane.

In this short communication we explain this feature. We also derive an equation for the curve and obtain the density of complex eigenvalues in terms of the well known objects of the conventional (Hermitian) disordered systems.

Most of our discussion will involve the lattice case which is technically simpler. Following [1], we will consider a one-dimensional non-Hermitian Anderson model whose eigenvalue equation reads as follows

$$-e^{\xi_{k-1}}\psi_{k-1} - e^{\eta_k}\psi_{k+1} + q_k\psi_k = z\psi_k, \;\; 1 \le k \le n \quad (1)$$
$$\psi_0 = \psi_n, \;\; \psi_1 = \psi_{n+1}. \quad (2)$$

The relevance of this non-Hermitian Anderson model to physics of vortex lines in superconductors is explained in [1]. In these works two particular cases of the model are discussed: (i) 'site-random' model with random q_k and with constant hopping elements given by $\eta_k \equiv h$ and $\xi_k \equiv -h$ (h is real); and (ii) 'random-hopping' model with $q_k \equiv 0$ and with random hopping elements given by $\eta_k = g^+ a_k$ and $\xi_k = g^- a_k$, where $g^\pm$ are some constants and the a_k are random variables.

Our basic assumptions about the coefficients in Eq. (1) are as follows: $\{(q_k, \xi_k, \eta_k)\}$ is a stationary sequence of random three-dimensional vectors such that $\langle \log(1 + |q_0|)\rangle$, $\langle \xi_0 \rangle$, $\langle \eta_0 \rangle$ are finite. The angle brackets denote averaging over the disorder.

2. We start our analysis with a standard transformation which is often used in the theory of differential and difference equations. Let us put $\psi_k = w_k\varphi_k$ in Eq. (1) and choose the weight w_k so that to make the resulting equation symmetric. For instance, if we set

$$w_0 = 1 \text{ and } w_k = e^{\frac{1}{2}\sum_{j=0}^{k-1}(\xi_j - \eta_j)} \text{ if } k \ge 1, \quad (3)$$

then this transformation reduces the eigenvalue problem (1)-(2) to the following one (we will use the notation $c_k = \exp[(\xi_k + \eta_k)/2]$ onwards)

$$-c_{k-1}\varphi_{k-1} - c_k\varphi_{k+1} + q_k\varphi_k = z\varphi_k \quad (4)$$
$$\varphi_{n+1} = w_{n+1}^{-1}w_1\varphi_1, \;\; \varphi_n = w_n^{-1}\varphi_0. \quad (5)$$

From now on, we will deal with Eqs. (4)–(5). [Obviously, the eigenvalues of (1)–(2) and (4)–(5) coincide.]

We can now introduce a "reference" symmetric eigenvalue problem which will be used in our analysis. This reference problem is specified by the same Eq. (4) and with the following boundary conditions (b.c.)

$$\varphi_{n+1} = 0, \;\; \varphi_0 = 0. \quad (6)$$

One can rewrite the eigenvalue problems (4),(5) and (4),(6) in the matrix form $\mathcal{H}\psi = z\psi$ and $H\varphi = z\varphi$, respectively. We use the calligraphic $\mathcal{H}$ for the non-symmetric problem (4),(5) and the usual italic H for the

symmetric problem (4),(6). H is a symmetric tridiagonal $n\times n$ matrix with the $\{q_k\}$ on the main diagonal and the $\{-c_k\}$ on the sub-diagonals. $\mathcal{H}$ is "almost" tridiagonal: the only non-zero elements of the difference $V=\mathcal{H}-H$ are $V_{1n}=-w_1^{-1}w_ne^{\xi_0}$ and $V_{n1}=-w_1w_n^{-1}e^{\eta_n}$.

Let $z_1,\ldots,z_n$ denote the eigenvalues of $\mathcal{H}$. Distribution of $\{z_j\}$ in the complex plane $z=x+iy$ is described by the measure

$$d\nu_n(z)\equiv d\nu_n(x,y)=\frac{1}{n}\sum_{j=1}^{n}\delta(x-x_j)\delta(y-y_j)dxdy.$$

Our goal now is to find the limit of $d\nu_n(z)$ when $n\to\infty$. To do this we will calculate the "electrostatic" potential $F(z)\equiv F(x,y)$ of the limit distribution of the eigenvalues of $\mathcal{H}$:

$$F(z)=\lim_{n\to\infty}F_n(z), \tag{7}$$

where

$$F_n(z)=\int_{\mathbf{C}}\log|z-z'|d\nu_n(z')=\frac{1}{n}\log|\det(\mathcal{H}-zI)| \tag{8}$$

is the potential of $d\nu_n(z)$. In Eq. (8) I is the $n\times n$ identity matrix.

According to the potential theory, the existence of the limit in Eq. (7) for almost all z implies the existence of the weak limit $d\nu(z)=\lim_{n\to\infty}d\nu_n(z)$. The limit measure is determined by the Poisson equation, i.e. $d\nu(z)\equiv d\nu(x,y)=1/(2\pi)\Delta F(x,y)\ dxdy$ [7,8].

The idea of using potentials to study eigenvalue distributions is not new and goes back to the 1960s at least, to studies of the eigenvalue distribution of Töplitz matrices [8]. In the context of random matrices this idea has been used since works [9,10].

We will prove below that with probability one the limit in Eq. (7) exists for almost all z and is given by

$$F(z)=\begin{cases} a & \text{if } \Phi(z)<a\\ \Phi(z) & \text{if } \Phi(z)>a,\end{cases} \tag{9}$$

where $a=\max(\langle\xi_0\rangle,\langle\eta_0\rangle)$ and $\Phi(z)$ is the potential of the limit distribution of the eigenvalues of H. In other words, $\Phi(z)=\lim_{n\to\infty}\Phi_n(z)$, where

$$\Phi_n(z)=\frac{1}{n}\log|\det(H-zI)|=\int_{-\infty}^{+\infty}\log|z-\lambda|dN_n(\lambda)$$

and $N_n(\lambda)$ is the number of eigenvalues of H in the interval $(-\infty,\lambda)$ divided by n. It is well known [11,12] that there exists a continuous non-random function $N(\lambda)$ such that with probability one $\lim_{n\to\infty}N_n(\lambda)=N(\lambda)$ and hence

$$\Phi(z)=\int_{-\infty}^{+\infty}\log|\lambda-z|dN(\lambda). \tag{10}$$

We will call the set which supports $d\nu(z)$ the limit spectrum of $\mathcal{H}$. $\Phi(z)$ as a function of $x=\operatorname{Re}z$ and $y=\operatorname{Im}z$ is harmonic everywhere in the complex plane except that part of the real axis which supports $dN(\lambda)$. Therefore, Eq. (9) says that the complex part of the limit spectrum is determined by the equation $\Phi(z)=a$. This equation defines a curve in the complex plane. We denote this curve by $\mathcal{L}$. The density of the eigenvalue distribution with respect to the arc-length measure ds on this curve equals to $(2\pi)^{-1}$ times jump in the normal derivative of F across the curve. Computing the derivative gives

$$\frac{d\nu}{ds}=\frac{1}{2\pi}\left|\int_{-\infty}^{+\infty}\frac{dN(\lambda)}{\lambda-z}\right|,\quad z\in\mathcal{L}. \tag{11}$$

$N(\lambda)$ is the classical integrated density of states associated with Eq. (4). It is very important that $\Phi(z)$ coincides, up to an additive constant, with the Lyapunov exponent $\gamma(z)$ of the same equation. Namely, the well known Thouless formula [14] states that $\Phi(z)=\gamma(z)+\frac{1}{2}(\langle\xi_0\rangle+\langle\eta_0\rangle)$ (for a rigorous proof of this formula and further references see e.g. [13]).

The spectral properties of $\mathcal{H}$ are, to a large extent, determined by the behaviour of $\gamma(z)$. This will be discussed later. Here we only mention that the equation $\Phi(z)=a$ which defines the curve $\mathcal{L}$ is equivalent to

$$\gamma(z)=\frac{1}{2}|\langle\xi_0\rangle-\langle\eta_0\rangle|. \tag{12}$$

Eq. (12) is more transparent. Loosely speaking, it says that a non-real z belongs to the limit spectrum of $\mathcal{H}$ only if there exists a solution φ of Eq. (4) such that $|\varphi_n|$ and w_n^{-1} are of the same order when $n\to\infty$ [compare this with Eq. (5)]. As an immediate consequence of Eq. (12) we obtain the following. If $\langle\xi_0\rangle=\langle\eta_0\rangle$ then there is no complex part in the limit spectrum of $\mathcal{H}$, i.e. $d\nu(z)$ is supported on the real axis only.

To derive Eq. (9) let us consider a non-real z such that $\Phi(z)\neq a$. Then $G=(H-zI)^{-1}$ is well defined and $\mathcal{H}-zI=(I+VG)(H-zI)$. Therefore

$$F_n(z)=\Phi_n(z)+\frac{1}{n}\log|d_n(z)|,$$

where $d_n(z)=\det(I+VG)$. Expanding this determinant yields

$$d_n(z)=[1+V_{1n}G_{n1}][1+V_{n1}G_{1n}]-V_{1n}V_{n1}G_{11}G_{nn},$$

where G_{jk} denotes the (j,k) matrix entry of G. To prove Eq. (9) it remains to find the $\lim_{n\to\infty}n^{-1}\log|d_n(z)|$.

Using $G_{1n}=G_{n1}=\prod_{j=1}^{n-1}c_k/\det(H_0-zI)$ one obtains

$$|V_{1n}G_{n1}|=\exp\{n[\langle\xi_0\rangle-\Phi(z)+r_n(z)+s_n]\},$$
$$|V_{n1}G_{1n}|=\exp\{n[\langle\eta_0\rangle-\Phi(z)+r_n(z)+t_n]\},$$

where $r_n(z)=\Phi(z)-\Phi_n(z)$, $s_n=n^{-1}\sum_{j=0}^{n-1}\xi_j-\langle\xi_0\rangle$, and $t_n=n^{-1}\sum_{j=1}^{n}\eta_j-\langle\eta_0\rangle$. As mentioned above, $\Phi_n(z)\to\Phi(z)$ and hence $r_n(z)\to 0$ when $n\to\infty$.

Also, because of the ergodic theorem $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 0$.

Consider first the case when $\Phi(z) < a$. Without loss of generality we may suppose that $a = \langle\eta_0\rangle$. Then $V_{n1}G_{1n}$ grows exponentially with n. On the other hand, $\Phi(z) > \langle\xi_0\rangle$ for every non-real z [15] and $V_{1n}G_{n1}$ tends to zero exponentially fast when $n\to\infty$. Besides, $V_{1n}V_{n1} = \exp[(\xi_0+\eta_n)/2]$ and the stationarity of the sequence $\{\eta_n\}$ implies that

$$\lim_{n\to\infty} n^{-1}\log|V_{1n}V_{n1}| = 0. \tag{13}$$

Therefore $\lim_{n\to\infty} n^{-1}\log|d_n(z)| = a - \Phi(z)$, hence $\lim_{n\to\infty} F_n(z) = a$ if $\Phi(z) < a$.

Consider now the case when $\Phi(z) > a$. Then both $V_{n1}G_{1n}$ and $V_{1n}G_{n1}$ are exponentially small if n is large. Therefore the "dangerous" term in $d_n(z)$ is $1 - V_{1n}V_{n1}G_{11}G_{nn}$. It can be shown [16] that $\alpha \le \arg G_{11} \le \pi - \alpha$, where α is strictly positive $(0 < \alpha < \pi/2)$ and depends only on z and on the realisation of $\{(q_k,\xi_k,\eta_k)\}$. One can deduce from this inequality that

$$|1 - V_{1n}V_{n1}G_{11}G_{nn}| \ge \sin\alpha.$$

This estimate together with Eq. (13) and the simple inequality $|G_{jj}| \le |\operatorname{Im} z|^{-1}$ prove that $n^{-1}\log|d_n(z)| \to 0$ when $n\to\infty$. Thus $\lim_{n\to\infty} F_n(z) = \Phi(z)$ if $\Phi(z) > a$.

Since the limit distribution of eigenvalues of $\mathcal{H}$ is one-dimensional, one can also use the conventional technique of Green's functions to obtain this limit distribution. Let z be non-real and $f_n(z) = n^{-1}\operatorname{tr}(\mathcal{H} - zI)^{-1}$, $g_n(z) = n^{-1}\operatorname{tr}(H - zI)^{-1}$, $f(z) = \lim_{n\to\infty} f_n(z)$, and $g(z) = \lim_{n\to\infty} g_n(z)$. Revising the above argument one can show that $f(z) = 0$ if $\Phi(z) < a$ and $f(z) = g(z)$ if $\Phi(z) > a$. This can also be shown by working directly with the Green's functions $f_n(z)$ and $g_n(z)$.

If $\{(q_k,\xi_k,\eta_k)\}$ is a sequence of independent identically distributed random vectors with finite variance (ξ_k and η_k are not necessarily independent), then more precise information on the large-n behaviour of $r_n(z)$, s_n, and t_n is available. Revising the above analysis and employing the limit-theorem techniques one can show in this case that

$$\rho_n \equiv \max_{z_j:|\operatorname{Im} z_j|>\varepsilon} \operatorname{dist}(z_j,\mathcal{L}) \tag{14}$$

is of the order $1/\sqrt{n}$ for any positive ε.

3. In this section, we consider another approach to the eigenvalue problem (4)–(5). It is based on the theory of Lyapunov exponents of products of random matrices. This approach explains the true reason for the appearance of the complex eigenvalues and gives an independent derivation of Eq. (12). It also provides more precise information about the finite-n behaviour of the eigenvalues of $\mathcal{H}$. Finally, it easily extends to the case of differential equations. It does not provide though the explicit form of the limit distribution derived above.

Let us introduce the usual transfer-matrices

$$A_k = \frac{1}{c_k}\begin{pmatrix} q_k - z & -c_{k-1} \\ c_k & 0 \end{pmatrix}. \tag{15}$$

Then the solution of Eq. (4) with initial data (φ_0,φ_1) can be written as

$$(\varphi_{k+1},\varphi_k)^T = S_k(z)(\varphi_1,\varphi_0)^T, \tag{16}$$

where $S_k(z) = A_kA_{k-1}\ldots A_1$. Thus the eigenvalue problem (4) – (5) reduces to

$$\left(B_nS_n(z) - w_{n+1}^{-1}I\right)(\varphi_1,\varphi_0)^T = 0, \tag{17}$$

where I is the 2×2 identity matrix and B_n is a 2×2 diagonal matrix, $B_n = \operatorname{diag}\{\exp[-\frac{1}{2}(\xi_0 - \eta_0)], \exp[-\frac{1}{2}(\xi_n - \eta_n)]\}$. In other words, the eigenvalues of $\mathcal{H}$ solve the equation $\det(B_nS_n(z) - w_{n+1}^{-1}I) = 0$. This equation is equivalent to $d_n(z) = 0$.

The Lyapunov exponent $\gamma(z)$ of $S_n(z)$ is defined as

$$\gamma(z) = \lim_{n\to\infty}\frac{1}{n}\log||S_n(z)||. \tag{18}$$

It is well known ([17], [11]) that under our basic assumptions on $\{(q_k,\xi_k,\eta_k)\}$ the limit on the right-hand side in Eq. (18) exists with probability one.

For every fixed z the right-hand side of Eq. (18) gives with probability 1 the same non-random number as the Thouless formula and the well known Furstenberg's formula [17]. To proceed, however, we have to study $\gamma(z)$ as a function of z when a (typical) realisation of $\omega \equiv \{(q_k,\xi_k,\eta_k)\}$ is fixed: $\gamma(z) = \gamma(z,\omega)$. Thus, we are faced with products of random matrices depending on a parameter. The latter were studied in [18].

Let Σ denote the spectrum the operator defined by the left hand side of Eq. (4) on $l_2(\mathbf{Z})$ [this set can equivalently be described as the support of the measure $dN(\lambda)$]. It was proved in [18] that with probability 1 the convergence in (18) is uniform in z if $|z| \le C$ and z does not belong to an ε-neighbourhood of Σ; here C is an arbitrary fixed large number and ε is an arbitrary strictly positive number. It can be deduced from this fact that: (a) if $\mu_n(z)$ is the largest by modulus eigenvalue of B_nS_n then $\lim_{n\to\infty} n^{-1}\log|\mu_n(z)| = \gamma(z)$ and the convergence here is again uniform in z in the same domain (clearly, the other eigenvalue is $e^{\xi_n+\eta_0}/\mu_n(z)$). (b) The following function $\bar\gamma(x) \equiv \gamma(x+i0)$ is now defined for almost every sequence ω simultaneously for all real x, $-\infty < x < \infty$; the latter limit exists because $\gamma(z)$ is monotone in y (the property which is obvious from the Thouless formula).

There is a subtlety here. Namely, we can consider $\gamma(x)$ defined by (18) for every fixed(!) x but not for all x simultaneously while the potential is fixed. Moreover, it turns out that if a typical sequence ω is fixed then there exists a "large" subset Θ_ω of Σ such that for all $x\in\Theta_\omega$ the limit in (18) either does not exist or does not coincide with $\bar\gamma(x)$ [18]. However the following statement which is very important for us holds true:

$$\bar\gamma(x) \ge \lim_{n\to\infty}\sup\frac{1}{n}\log||S_n(x)|| \equiv \tilde\gamma(x) \tag{19}$$

A more detailed discussion of properties (a), (b), and (19) along with a very simple proof of the mentioned above uniform convergence in (18) will be provided in [16].

Combining property (a) with the statement that $\lim_{n\to\infty} n^{-1}\log w_{n+1} = (\langle\xi_0\rangle - \langle\eta_0\rangle)/2$ one concludes that all non-real solutions of Eq. (17) are asymptotically (as $n\to\infty$) attracted to the curve $\mathcal{L}$ given by Eq. (12). Moreover, it follows from the uniform convergence of $\mu_n(z)$ that $\lim_{n\to\infty}\rho_n = 0$, where ρ_n is defined by (14).

It is worth noticing that Eq. (18) can very rarely be used for calculation of the Lyapunov exponent $\gamma(z)$. In order to make use of Eq. (12) one has to compute $\gamma(z)$. One way to do this is provided by the Thouless formula. The other one is the well known Furstenberg's formula [17]. To a large extent, both approaches are equivalent and in order to use any of these formulae one has to find an invariant measure of a certain transformation for which purpose an integral equation has to be solved. Though in general calculations of this sort are difficult, for some distributions the explicit expression for $\gamma(z)$ and $N(\lambda)$ can be found; one of these examples is discussed below.

The properties of the two functions, $\bar{\gamma}(x)$ and $\tilde{\gamma}(x)$, are crucial for further analysis. Suppose for simplicity that in addition to our basic assumptions (q_n, ξ_n, η_n) form a sequence of independent identically distributed random vectors. Then it is easy to show that for sufficiently large $|x|$

$$\log|x| - C_1 \le \bar{\gamma}(x) \le \log|x| + C_1, \tag{20}$$

where C_1 depends only on the distribution of (q_0, ξ_0, η_0). Then, obviously, there exist C such that all solutions of Eq. (12) belong to a circle $|z|\le C$.

The other useful property of $\bar{\gamma}(x)$ is as follows: if in addition to the above assumptions q_0 takes at least two different values then $\bar{\gamma}(x) > 0$ is strictly positive (see [17,12,13]). Finally, under these conditions, $\bar{\gamma}(x)$ is a continuous function of x (see, e.g. [18]).

We are now in a position to describe the curve $\mathcal{L}$. Consider the inequality

$$\bar{\gamma}(x) \le \frac{1}{2}|\langle\xi_0\rangle - \langle\eta_0\rangle| \tag{21}$$

Because of the continuity of $\bar{\gamma}(x)$, the solution of this inequality is given by a union of disjoint intervals: $\cup_j [a_j, a'_j]$ with $a_j < a'_j$. Next, for every $x\in[a_j, a'_j]$, consider a positive $y = y_j(x)$ such that the pair (x, y) solves Eq. (12). This solution exists and is unique because if x is fixed then $\gamma(x+iy)$ is strictly monotone continuous function of positive y and $\lim_{y\to+\infty}\gamma(x+iy)=\infty$. The curve $\mathcal{L}$ (the solution of Eq. (12)) is a union of disconnected contours: $\mathcal{L} = \cup_j\mathcal{L}_j$. Each contour $\mathcal{L}_j$ consists of two symmetric smooth arcs $y = y_j(x)$ and $y = -y_j(x)$, where $a_j \le x \le a'_j$; the points $(a_j, 0)$ and $(a'_j, 0)$ are the end-points of these arcs. [We notice that it is easy to construct examples with a prescribed finite number of contours. In general, there is no obvious reason for the number of contours to be finite for an arbitrary distribution of $(\xi_0,\ \eta_0,\ q_0)$.]

Our next goal is to describe the behaviour of the real eigenvalues of Eqs. (4)–(5). It turns out that this behaviour is governed by the following remarkable property of $\tilde{\gamma}(x)$: this function is upper semi-continuous at each point x where $\tilde{\gamma}(x) = \bar{\gamma}(x)$ (this equality holds for almost every x; remember that $\tilde{\gamma}(x)$ depends strongly on ω [18])

It can be deduced from this property [16] that for every strictly positive ε the spectrum of $\mathcal{H}$ lies outside a domain surrounded by the following strip:

$$\mathcal{D}_{j,\varepsilon} \equiv \{z\in\mathbf{C} : \mathrm{dist}(|z|, \mathcal{L}_j) \le \varepsilon\}$$

if n is large enough. In other words, with probability 1 the spectrum of $\mathcal{H}$ is wiped out from the interior of every contour $\mathcal{L}_j$ as $n\to\infty$.

Finally, here is a description of the limit spectrum of Eqs. (4)–(5). Let Σ be the same as above (i.e. the spectrum of the reference symmetric problem). Then the set

$$\mathcal{L}\cup\left\{\Sigma\setminus\cup_j\,(a_j, a'_j)\right\} \tag{22}$$

is the limit spectrum of $\mathcal{H}$.

4. We start our discussion with a remark on the spectrum of the limit operator $\hat{\mathcal{H}}$ defined by the left-hand side of Eq. (1) on $l_2(\mathbf{Z})$. For simplicity we suppose that all coefficients in Eq. (1) are bounded. It turns out that for a wide class of distributions of $\{(\xi_k, \eta_k, q_k)\}$ the spectrum of $\hat{\mathcal{H}}$ is a two-dimensional subset in the complex plane and the limit spectrum of $\mathcal{H}$ given by Eq. (22) is embedded into this set. This phenomenon seems to be surprising because $\mathcal{H}\varphi$ converges to $\hat{\mathcal{H}}\varphi$ when $n\to\infty$ (n is the dimension of $\mathcal{H}$) for every $\varphi\in l_2(\mathbf{Z})$.

Since the problem is non-Hermitian, the spectrum may depend on the choice of boundary conditions (b.c.). Indeed, all the eigenvalues of Eq. (1) with the Dirichlet b.c., i.e. when $\psi_{n+1} = \psi_0 = 0$, are real. It is remarkable, however, that the boundary conditions of the form $(\psi_{n+1}, \psi_n)^T = B(\psi_1, \psi_0)^T$, where B is a fixed real *non-degenerate* 2×2 matrix, lead to the same Eq. (12) (regardless of the choice of B) in the limit $n\to\infty$. For diagonal B this fact can be readily seen from our proof of Eq. (9). The general case of a non-degenerate B is less transparent and our proof of Eq. (12) in this case [16] relies on the above mentioned properties of products of random matrices [18].

The reader might have noticed that our derivation of Eq. (9) is based only on the fact of existence of $N(\lambda)$. (Under our basic assumptions the existence of $N(\lambda)$ is ensured by the stationarity of the coefficients.) Thus one can easily extend our argument to other classes of coefficients.

For instance, instead of a random sequence $\{(\xi_k, \eta_k, q_k)\}$ one can take a periodic one (i.e. the q_k, ξ_k, and η_k all have a common period). Then Eq. (9) holds with the following obvious change. In the periodic

case the angle brackets denote averaging over the period. The complex part of the limit spectrum is described by the same equation (12) in which $\gamma(z)$ is the Lyapunov exponent of the symmetric reference equation (4) whose coefficients are now periodic.

The geometry of the limit spectrum of $\mathcal{H}$ in the periodic case follows two patterns. If $\langle\xi\rangle = \langle\eta\rangle$ then the limit spectrum is real and coincides with Σ, the spectrum of the symmetric reference equation. If $\langle\xi\rangle \neq \langle\eta\rangle$ the limit spectrum is purely complex, i.e. the limit distribution of eigenvalues is supported on the curve defined by Eq. (12). It is worth mentioning that in the periodic case this curve is a union of a finite number of analytic contours. Indeed, because $\gamma(z) = 0$ on Σ in the periodic case, the above mentioned arcs join up smoothly. In either case (real or complex spectrum) the corresponding eigenfunctions are extended. The case when $\{(\xi_k, \eta_k, q_k)\}$ is quasi-periodic is much more delicate and deserves a separate study.

The rest of our discussion is based on known results on Hermitian disordered systems and the formulae derived above. From now on we assume that the $\{q_k\}$ is a sequence of independent identically distributed random variables which is also independent of the $\{c_k\}$. It is instructive to consider first an exactly solvable model.

Let $c_k \equiv 1$ and q_k are Cauchy distributed, i.e. $\mathrm{Prob}\{q_k \in \Delta\} = \pi^{-1}\int_\Delta dq\; b/(q^2+b^2)$. In this special case an explicit expression for $N(\lambda)$ is known (see, e.g. [13]). Using this expression and the Thouless formula one obtains that

$$4\cosh\gamma(z) = \sqrt{(x+2)^2 + (b+|y|)^2} + \sqrt{(x-2)^2 + (b+|y|)^2}, \tag{23}$$

where $x \equiv \mathrm{Re}\, z$ and $y \equiv \mathrm{Im}\, z$. One can use this result to obtain the limit spectrum of a family of non-Hermitian operators $\mathcal{H}$ (1)–(2). Suppose that in Eq. (1) $\eta_k = -\xi_k$ for every k (e.g. one can take $\eta_k \equiv h$ and $\xi_k \equiv -h$ as in the site-random model). Then in the reference equation $c_k \equiv 1$ and by straightforward computations involving Eqs. (12) and (23) one finds that the complex part of the limit spectrum of $\mathcal{H}$ consists of the two arcs

$$y(x) = \pm\left[\sqrt{\frac{(K^2-4)(K^2-x^2)}{K^2}} - b\right], -x_b \le x \le x_b, \tag{24}$$

where $K = 2\cosh\langle\eta_0\rangle$ and x_b is determined by the condition $y(x_b) = 0$. The real eigenvalues of $\mathcal{H}$ are distributed (in the limit $n \to \infty$) in the intervals $(-\infty, -x_b)$ and $(x_b, +\infty)$ with density equal to the density of eigenvalues of Eq. (4) in these intervals.

The b-dependence of the complex part of the limit spectrum is remarkably simple. If $b = 0$ the arcs (24) form the ellipse $x^2/K^2 + y^2/(K^2-4) = 1$. As b increases each of the arcs moves (by translation in the y-direction) towards the real axis and reduces in length. At $b = \sqrt{K^2-4}$ the arcs (hence the complex part of the limit spectrum) disappear. In other words, if $K < K_{cr} = \sqrt{4+b^2}$ Eq. (12) cannot be solved and the limit spectrum of $\mathcal{H}$ is real. If $K > K_{cr}$ then the limit spectrum of $\mathcal{H}$ has always two branching points from which the complex branches grow.

We should mention another example of the reference eigenvalue problem (4) amenable to analytic treatment. In this example the c_k are independent exponentially distributed random variables and $q_k \equiv 0$. Thus this example is related to the random-hopping non-Hermitian Anderson model. The known expression for $\gamma(z)$ [19] is not that simple as in the previous case and in order to analyse Eq. (12) one has to employ certain approximations. This analysis goes beyond the scope of the present publication.

The next part of our discussion involves general analysis of Eqs. (9) and (12). Denote $2g \equiv \langle\eta_0\rangle - \langle\xi_0\rangle$. As mentioned above the limit spectrum of $\mathcal{H}$ is entirely real if $g = 0$. First, we want to show that in the case when $q_k \equiv 0$ (random hopping model) the limit spectrum has a non-trivial complex part for every non-zero g. Indeed, in this case the transfer matrices A_k (15) have zero diagonal elements when $z = 0$. Hence their products and the norm of $S_n(0)$ can be easily computed (see, e.g. [13]). From this computation and the stationarity of the c_k it follows easily that $\gamma(0) = 0$. Therefore, if $g \neq 0$ the solutions to the equation $\gamma(z) = |g|$ constitute a non-trivial (non-real) curve.

This is not the case if the distribution of q_0 (hence of all q_k) is non-degenerate (i.e. q_0 takes at least two different values). Then there exists $g_{cr}^{(1)} > 0$ such that for all $|g| \le g_{cr}^{(1)}$ the limit spectrum of $\mathcal{H}$ has no complex part. Indeed, if the q_k are non-degenerate $\bar\gamma(x) \equiv \gamma(x+i0)$ is strictly positive on the limit spectrum Σ of the reference eigenvalue problem (4),(5). The continuity of $\bar\gamma(z)$, Eq. (20) and the inequality $\gamma(x+iy) \ge \bar\gamma(x)$ imply that

$$g_{cr}^{(1)} \equiv \min_{x\in\Sigma} \bar\gamma(x) > 0.$$

If in addition the coefficients in Eq. (4) are bounded, i.e. $c_k^2 + q_k^2 \le C$ for all k, then Σ is a bounded set. Therefore

$$g_{cr}^{(2)} \equiv \max_{x\in\Sigma} \bar\gamma(x)$$

is finite. If $|g| \ge g_{cr}^{(2)}$ the inequality (21) is satisfied for every point of Σ. Hence the limit spectrum is purely complex in this case. If either c_0 or q_0 takes arbitrary large values with non-zero probability, then Σ is an unbounded set and in view of Eq. (20) $g_{cr}^{(2)} = +\infty$.

In summary, if $|g| \le g_{cr}^{(1)}$ the limit spectrum of $\mathcal{H}$ is entirely real, if $|g| \ge g_{cr}^{(2)}$ the limit spectrum is entirely complex and if $g_{cr}^{(1)} < |g| < g_{cr}^{(2)}$ the limit spectrum has real and complex parts. In the latter case the branching points from which the complex branches grow out of the real eigenvalues are determined by $\bar\gamma(x) = |g|$.

The limit distribution of the real eigenvalues of $\mathcal{H}$ is described by $N(\lambda)$, the integrated density of states of the reference symmetric equation. The limit distribution of the non-real eigenvalues of $\mathcal{H}$ is described by Eq.

(11). It should be noticed that the density of the non-real eigenvalues given by Eq. (11) is analytic everywhere on $\mathcal{L}$ except the (real) end-points of the arcs. (If the limit spectrum is entirely complex then this density is analytic everywhere.) The behaviour of the density of the non-real eigenvalues near an end-point x_j of an arc depends on the regularity properties of $N(\lambda)$ at this point. If the density of states $N'(\lambda)$ of the reference equation is smooth in a neighbourhood of the point $\lambda = x_j$ then the density of the limit distribution of the non-real eigenvalues of $\mathcal{H}$ has a finite limit as z approaches x_j along the arc. Also, in this case the tangent to the arc exists at x_j and is not vertical. In other words, if $N'(\lambda)$ is smooth in a neighbourhood of a branching point $\lambda = x_j$ the complex branches of the limit spectrum grow out of x_j linearly. This may not be the case if $N'(\lambda)$ is not smooth.

After this work was completed we learned about recent preprints [20] addressing similar problems.

[1] N.Hatano and D.R.Nelson, Phys.Rev.Lett. **77**, 570 (1996); cond-mat/9705290.

[2] J. Miller and Z.J. Wang, Phys.Rev.Lett. **76**, 1461 (1996); J.T. Chalker and Z.J. Wang, cond-mat/9704198.

[3] M.A. Stephanov, Phys.Rev.Lett. **76**, 4472 (1996); J.J.M. Verbaarschot, Nucl. Phys. Proc. Suppl. **53**, 88 (1997).

[4] Y.V.Fyodorov and H.-J.Sommers, JETF Lett. **63**, 1026 (1986); J.Math.Phys. **38**, 1918 (1997).

[5] K.B.Efetov, Phys.Rev.Lett. **79** 491 (1997).

[6] R.A.Janik, M.A.Nowak, G.Papp and I.Zahed, cond-mat/9612240; J.Feinberg and A.Zee, cond-mat/9703087; Y.V.Fyodorov, B.A.Khoruzhenko and H.-J.Sommers, Phys.Rev.Lett. **79**, 557 (1997).

[7] L.Hörmander, Notions of Convexity (Birkhäuser, Boston, 1994).

[8] H.Widom, Operator Theory: Adv. Appl. **71**, 1 (1994)

[9] V.Girko, Theor.Prob.Appl. **29**,694(1985); **30**,677(1986).

[10] H.-J.Sommers, A.Crisanti, H.Sompolinsky, and Y.Stein, Phys.Rev.Lett. **60**, 1895 (1988).

[11] P.Bougerol and J. Lacroix, Products of Random Matrices with Applications to Random Schrödinger Operators (Birkhäuser, Boston, 1985).

[12] R.Carmona and J. Lacroix, Spectral Theory of Random Schrödinger Operators (Birkhäuser, Boston, 1990).

[13] L.A.Pastur and A.L.Figotin, Spectra of Random and Almost-Periodic Operators (Springer, Berlin, 1992).

[14] D.J. Thouless, J.Phys.C **5**, 77 (1972).

[15] This inequality can be inferred from the Thouless formula and the fact that $\gamma(z) \geq 0$ for real z.

[16] I.Goldsheid and B.A. Khoruzhenko, in preparation.

[17] H.Furstenberg, Trans.Amer.Math.Soc., **108**, 377 (1963).

[18] I. Goldsheid, Dokl. Akad. Nauk. SSSR, **224**, 1248 (1975); in Adv. in Prob. **8**, p. 239, eds. Ya. G. Sinai and R. Dobrushin (Dekker, N.Y., 1980).

[19] F.Dyson, Phys.Rev. **92**, 1331 (1953).

[20] R.A.Janik, M.A.Nowak, G.Papp and I.Zahed, cond-mat/9705098; P.W. Brower, P.G. Silvestrov, C.W.J. Beenakker, cond-mat/9705186; J.Feinberg and A.Zee, cond-mat/9706218;